



I.1 A BRIEF PREVIEW OF CALCULUS: TANGENT LINES AND THE LENGTH OF A CURVE

In this section, we approach the boundary between precalculus mathematics and the calculus by investigating several important problems requiring the use of calculus. Recall that the slope of a straight line is the change in y divided by the change in x . This fraction is the same regardless of which two points you use to compute the slope. For example, the points $(0, 1)$, $(1, 4)$, and $(3, 10)$ all lie on the line $y = 3x + 1$. The slope of 3 can be obtained from any two of the points. For instance,

$$m = \frac{4 - 1}{1 - 0} = 3 \quad \text{or} \quad m = \frac{10 - 1}{3 - 0} = 3.$$

In the calculus, we generalize this problem to find the slope of a *curve* at a point. For instance, suppose we wanted to find the slope of the curve $y = x^2 + 1$ at the point $(1, 2)$. You might think of picking a second point on the parabola, say $(2, 5)$. The slope of the line through these two points (called a **secant line**; see Figure 1.2a) is easy enough to compute. We have

$$m_{\text{sec}} = \frac{5 - 2}{2 - 1} = 3.$$

However, using the points $(0, 1)$ and $(1, 2)$, we get a different slope (see Figure 1.2b):

$$m_{\text{sec}} = \frac{2 - 1}{1 - 0} = 1.$$

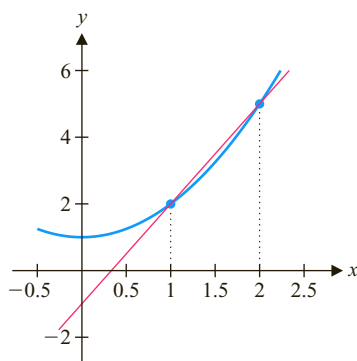


FIGURE 1.2a

Secant line, slope = 3

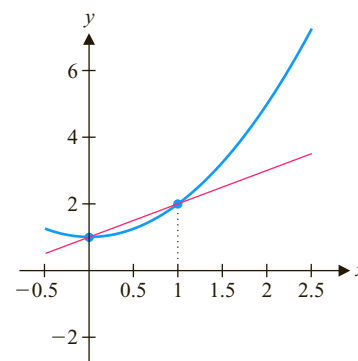


FIGURE 1.2b

Secant line, slope = 1

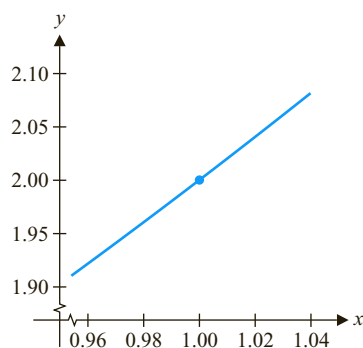


FIGURE 1.3

$y = x^2 + 1$

For curves other than straight lines, the slopes of secant lines joining different points are generally *not* the same, as seen in Figures 1.2a and 1.2b.

If you get different slopes using different pairs of points, then what exactly does it mean for a curve to have a slope at a point? The answer can be visualized by graphically zooming in on the specified point. Take the graph of $y = x^2 + 1$ and zoom in tight on the point $(1, 2)$. You should get a graph something like the one in Figure 1.3. The graph looks very much like a straight line. In fact, the more you zoom in, the straighter the curve appears to be and the less it matters which two points are used to compute a slope. So, here's the strategy: pick several points on the parabola, each closer to the point $(1, 2)$ than the previous one. Compute the slopes of the lines through $(1, 2)$ and each of the points. The closer the second point gets to $(1, 2)$, the closer the computed slope is to the answer you seek.

For example, the point $(1.5, 3.25)$ is on the parabola fairly close to $(1, 2)$. The slope of the line joining these points is

$$m_{\text{sec}} = \frac{3.25 - 2}{1.5 - 1} = 2.5.$$

The point $(1.1, 2.21)$ is even closer to $(1, 2)$. The slope of the secant line joining these two points is

$$m_{\text{sec}} = \frac{2.21 - 2}{1.1 - 1} = 2.1.$$

Continuing in this way, observe that the point $(1.01, 2.0201)$ is closer yet to the point $(1, 2)$. The slope of the secant lines through these points is

$$m_{\text{sec}} = \frac{2.0201 - 2}{1.01 - 1} = 2.01.$$

The slopes of the secant lines $(2.5, 2.1, 2.01)$ are getting closer and closer to the slope of the parabola at the point $(1, 2)$. Based on these calculations, it seems reasonable to say that the slope of the curve is approximately 2.

Example 1.1 takes our introductory example just a bit further.

EXAMPLE 1.1 Estimating the Slope of a Curve

Estimate the slope of $y = x^2 + 1$ at $x = 1$.

Solution We focus on the point whose coordinates are $x = 1$ and $y = 1^2 + 1 = 2$. To estimate the slope, choose a sequence of points near $(1, 2)$ and compute the slopes of the secant lines joining those points with $(1, 2)$. (We showed sample secant lines in Figures 1.2a and 1.2b.) Choosing points with $x > 1$ (x -values of 2, 1.1 and 1.01) and points with $x < 1$ (x -values of 0, 0.9 and 0.99), we compute the corresponding y -values using $y = x^2 + 1$ and get the slopes shown in the following table.

Second Point	m_{sec}
$(2, 5)$	$\frac{5 - 2}{2 - 1} = 3$
$(1.1, 2.21)$	$\frac{2.21 - 2}{1.1 - 1} = 2.1$
$(1.01, 2.0201)$	$\frac{2.0201 - 2}{1.01 - 1} = 2.01$

Second Point	m_{sec}
$(0, 1)$	$\frac{1 - 2}{0 - 1} = 1$
$(0.9, 1.81)$	$\frac{1.81 - 2}{0.9 - 1} = 1.9$
$(0.99, 1.9801)$	$\frac{1.9801 - 2}{0.99 - 1} = 1.99$

Observe that in both columns, as the second point gets closer to $(1, 2)$, the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the curve at the point $(1, 2)$ is then 2. ■

In Chapter 2, we develop a powerful technique for computing such slopes exactly (and easily). Note what distinguishes the calculus problem from the corresponding algebra problem. The calculus problem involves a process we call a *limit*. While we presently can

only estimate the slope of a curve using a sequence of approximations, the limit allows us to compute the slope exactly.

EXAMPLE 1.2 Estimating the Slope of a Curve

Estimate the slope of $y = \sin x$ at $x = 0$.

Solution This turns out to be a very important problem, one that we will return to later. For now, choose a sequence of points near $(0, 0)$ and compute the slopes of the secant lines joining those points with $(0, 0)$. The following table shows one set of choices.

Second Point	m_{sec}
$(1, \sin 1)$	0.84147
$(0.1, \sin 0.1)$	0.99833
$(0.01, \sin 0.01)$	0.99998

Second Point	m_{sec}
$(-1, \sin(-1))$	0.84147
$(-0.1, \sin(-0.1))$	0.99833
$(-0.01, \sin(-0.01))$	0.99998

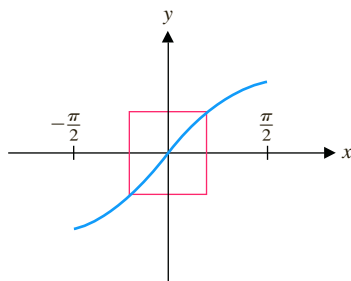


FIGURE 1.4
 $y = \sin x$

Note that as the second point gets closer and closer to $(0, 0)$, the slope of the secant line (m_{sec}) appears to get closer and closer to 1. A good estimate of the slope of the curve at the point $(0, 0)$ would then appear to be 1. Although we presently have no way of computing the slope exactly, this is consistent with the graph of $y = \sin x$ in Figure 1.4. Note that near $(0, 0)$, the graph resembles that of $y = x$, a straight line of slope 1. ■

A second problem requiring the power of calculus is that of computing distance along a curved path. While this problem is of less significance than our first example (both historically and in the development of the calculus), it provides a good indication of the need for mathematics beyond simple algebra. You should pay special attention to the similarities between the development of this problem and our earlier work with slope.

Recall that the (straight-line) distance between two points (x_1, y_1) and (x_2, y_2) is

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For instance, the distance between the points $(0, 1)$ and $(3, 4)$ is

$$d\{(0, 1), (3, 4)\} = \sqrt{(3 - 0)^2 + (4 - 1)^2} = 3\sqrt{2} \approx 4.24264.$$

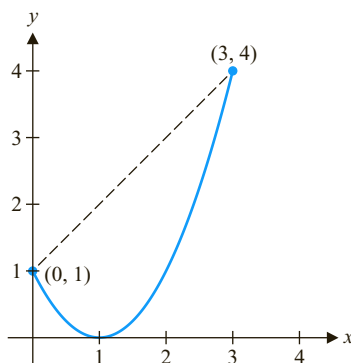


FIGURE 1.5a
 $y = (x - 1)^2$

However, this is not the only way we might want to compute the distance between these two points. For example, suppose that you needed to drive a car from $(0, 1)$ to $(3, 4)$ along a road that follows the curve $y = (x - 1)^2$ (see Figure 1.5a). In this case, you don't care about the straight-line distance connecting the two points, but only about how far you must drive along the curve (the *length* of the curve or *arc length*).

Notice that the distance along the curve must be greater than $3\sqrt{2}$ (the straight-line distance). Taking a cue from the slope problem, we can formulate a strategy for obtaining a sequence of increasingly accurate approximations. Instead of using just one line segment to get the approximation of $3\sqrt{2}$, we could use two line segments, as in Figure 1.5b. Notice that the sum of the lengths of the two line segments appears to be a much better approximation to the actual length of the curve than the straight-line distance used previously. This

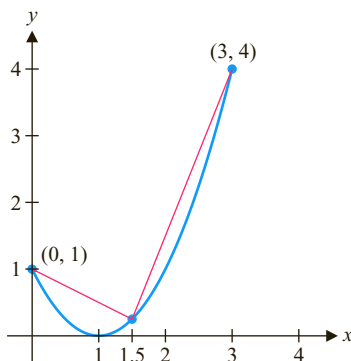


FIGURE 1.5b
Two line segments

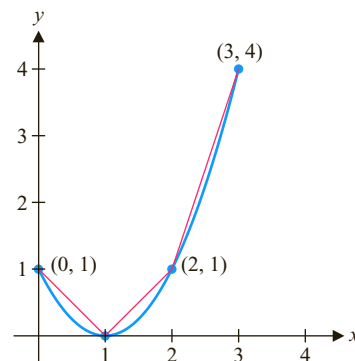


FIGURE 1.5c
Three line segments

distance is

$$\begin{aligned} d_2 &= d\{(0, 1), (1.5, 0.25)\} + d\{(1.5, 0.25), (3, 4)\} \\ &= \sqrt{(1.5 - 0)^2 + (0.25 - 1)^2} + \sqrt{(3 - 1.5)^2 + (4 - 0.25)^2} \approx 5.71592. \end{aligned}$$

You're probably way ahead of us by now. If approximating the length of the curve with two line segments gives an improved approximation, why not use three or four or more? Using the three line segments indicated in Figure 1.5c, we get the further improved approximation

$$\begin{aligned} d_3 &= d\{(0, 1), (1, 0)\} + d\{(1, 0), (2, 1)\} + d\{(2, 1), (3, 4)\} \\ &= \sqrt{(1 - 0)^2 + (0 - 1)^2} + \sqrt{(2 - 1)^2 + (1 - 0)^2} + \sqrt{(3 - 2)^2 + (4 - 1)^2} \\ &= 2\sqrt{2} + \sqrt{10} \approx 5.99070. \end{aligned}$$

No. of Segments	Distance
1	4.24264
2	5.71592
3	5.99070
4	6.03562
5	6.06906
6	6.08713
7	6.09711

Note that the more line segments we use, the better the approximation appears to be. This process will become much less tedious with the development of the definite integral in Chapter 4. For now we list a number of these successively better approximations (produced using points on the curve with evenly spaced x -coordinates) in the table found in the margin. The table suggests that the length of the curve is approximately 6.1 (quite far from the straight-line distance of 4.2). If we continued this process using more and more line segments, the sum of their lengths would approach the actual length of the curve (about 6.126). As in the problem of computing the slope of a curve, the exact arc length is obtained as a limit.

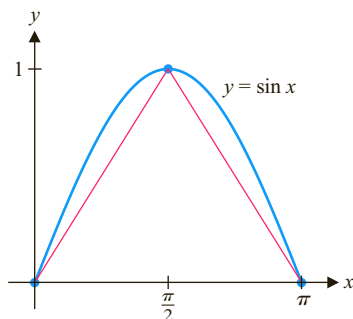


FIGURE 1.6a
Approximating the curve with two line segments

EXAMPLE 1.3 Estimating the Arc Length of a Curve

Estimate the arc length of the curve $y = \sin x$ for $0 \leq x \leq \pi$ (see Figure 1.6a).

Solution The endpoints of the curve on this interval are $(0, 0)$ and $(\pi, 0)$. The distance between these points is $d_1 = \pi$. The point on the graph of $y = \sin x$ corresponding to the midpoint of the interval $[0, \pi]$ is $(\pi/2, 1)$. The distance from $(0, 0)$ to $(\pi/2, 1)$ plus the distance from $(\pi/2, 1)$ to $(\pi, 0)$ (illustrated in Figure 1.6a) is

$$d_2 = \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} + \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \approx 3.7242.$$

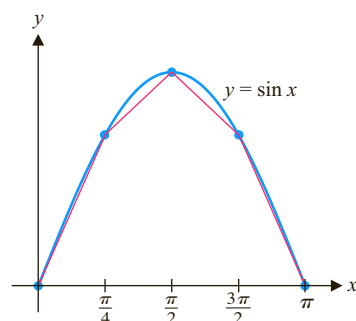


FIGURE 1.6b

Approximating the curve with four line segments

Number of Line Segments	Sum of Lengths
8	3.8125
16	3.8183
32	3.8197
64	3.8201

Using the five points $(0, 0)$, $(\pi/4, 1/\sqrt{2})$, $(\pi/2, 1)$, $(3\pi/4, 1/\sqrt{2})$, and $(\pi, 0)$ (i.e., four line segments, as indicated in Figure 1.6b), the sum of the lengths of these line segments is

$$d_4 = 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}} + 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} \approx 3.7901.$$

Using nine points (i.e., eight line segments), you need a good calculator and some patience to compute the distance of 3.8125. A table showing further approximations is given in the margin. At this stage, it would be reasonable to estimate the length of the sine curve on the interval $[0, \pi]$ as slightly more than 3.8. ■

BEYOND FORMULAS

In the process of estimating both the slope of a curve and the length of a curve, we make some reasonably obvious (straight-line) approximations and then systematically improve on those approximations. In each case, the shorter the line segments are, the closer the approximations are to the desired value. The essence of this is the concept of *limit*, which separates precalculus mathematics from the calculus. At first glance, this limit idea might seem of little practical importance, since in our examples we never compute the exact solution. In the chapters to come, we will find remarkably simple shortcuts to exact answers. Can you think of ways to find the exact slope in example 1.1?

EXERCISES 1.1

WRITING EXERCISES

1. Explain why each approximation of arc length in example 1.3 is less than the actual arc length.
2. To estimate the slope of $f(x) = x^2 + 1$ at $x = 1$, you would compute the slopes of various secant lines. Note that $y = x^2 + 1$ curves up. Explain why the secant line connecting $(1, 2)$ and $(1.1, 2.21)$ will have slope greater than the slope of the tangent line. Discuss how the slope of the secant line between $(1, 2)$ and $(0.9, 1.81)$ compares to the slope of the tangent line.

In exercises 1–12, estimate the slope (as in example 1.1) of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 + 1, a = 1$
2. $f(x) = x^2 + 1, a = 2$
3. $f(x) = \cos x, a = 0$
4. $f(x) = \cos x, a = \pi/2$
5. $f(x) = x^3 + 2, a = 1$
6. $f(x) = x^3 + 2, a = 2$
7. $f(x) = \sqrt{x+1}, a = 0$
8. $f(x) = \sqrt{x+1}, a = 3$

9. $f(x) = e^x, a = 0$
10. $f(x) = e^x, a = 1$
11. $f(x) = \ln x, a = 1$
12. $f(x) = \ln x, a = 2$

In exercises 13–20, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n 's and conjecture the actual length of the curve.

13. $f(x) = x^2 + 1, 0 \leq x \leq 2$
14. $f(x) = x^3 + 2, 0 \leq x \leq 1$
15. $f(x) = \cos x, 0 \leq x \leq \pi/2$
16. $f(x) = \sin x, 0 \leq x \leq \pi/2$
17. $f(x) = \sqrt{x+1}, 0 \leq x \leq 3$
18. $f(x) = 1/x, 1 \leq x \leq 2$
19. $f(x) = x^2 + 1, -2 \leq x \leq 2$
20. $f(x) = x^3 + 2, -1 \leq x \leq 1$

21. An important problem in calculus is finding the area of a region. Sketch the parabola $y = 1 - x^2$ and shade in the region above the x -axis between $x = -1$ and $x = 1$. Then sketch in the following rectangles: (1) height $f(-\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = -1$ to $x = -\frac{1}{2}$. (2) height $f(-\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = -\frac{1}{2}$ to $x = 0$. (3) height $f(\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = 0$ to $x = \frac{1}{2}$. (4) height $f(\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = \frac{1}{2}$ to $x = 1$. Compute the sum of the areas of the rectangles. Based on your sketch, does this give you a good approximation of the area under the parabola?
22. To improve the approximation of exercise 21, divide the interval $[-1, 1]$ into 8 pieces and construct a rectangle of the appropriate height on each subinterval. Compared to the approximation in exercise 21, explain why you would expect this to be a better approximation of the actual area under the parabola.
23. Use a computer or calculator to compute an approximation of the area in exercise 21 using (a) 16 rectangles, (b) 32 rectangles, (c) 64 rectangles. Use these calculations to conjecture the exact value of the area under the parabola.
24. Use the technique of exercises 21–23 to estimate the area below $y = \sin x$ and above the x -axis between $x = 0$ and $x = \pi$.
25. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x -axis between $x = 0$ and $x = 1$.
26. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x -axis between $x = 0$ and $x = 2$.



EXPLORATORY EXERCISE

1. Several central concepts of calculus have been introduced in this section. An important aspect of our future development of calculus is to derive simple techniques for computing quantities such as slope and arc length. In this exercise, you will learn how to directly compute the slope of a curve at a point. Suppose you want the slope of $y = x^2$ at $x = 1$. You could start by computing slopes of secant lines connecting the point $(1, 1)$ with nearby points. Suppose the nearby point has x -coordinate $1 + h$, where h is a small (positive or negative) number. Explain why the corresponding y -coordinate is $(1 + h)^2$. Show that the slope of the secant line is $\frac{(1 + h)^2 - 1}{1 + h - 1} = 2 + h$. As h gets closer and closer to 0, this slope better approximates the slope of the tangent line. Letting h approach 0, show that the slope of the tangent line equals 2. In a similar way, show that the slope of $y = x^2$ at $x = 2$ is 4 and find the slope of $y = x^2$ at $x = 3$. Based on your answers, conjecture a formula for the slope of $y = x^2$ at $x = a$, for any unspecified value of a .



1.2 THE CONCEPT OF LIMIT

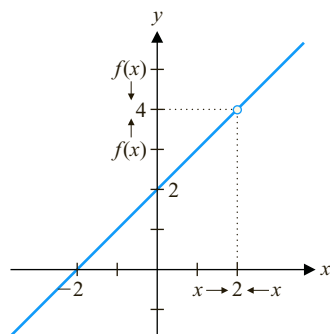


FIGURE 1.7a

$$y = \frac{x^2 - 4}{x - 2}$$

In this section, we develop the notion of limit using some common language and illustrate the idea with some simple examples. The notion turns out to be a rather subtle one, easy to think of intuitively, but a bit harder to pin down in precise terms. We present the precise definition of limit in section 1.6. There, we carefully define limits in considerable detail. The more informal notion of limit that we introduce and work with here and in sections 1.3, 1.4 and 1.5 is adequate for most purposes.

As a start, consider the functions

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5}{x - 2}.$$

Notice that both functions are undefined at $x = 2$. So, what does this mean, beyond saying that you cannot substitute 2 for x ? We often find important clues about the behavior of a function from a graph (see Figures 1.7a and 1.7b).

Notice that the graphs of these two functions look quite different in the vicinity of $x = 2$. Although we can't say anything about the value of these functions at $x = 2$ (since this is outside the domain of both functions), we can examine their behavior in the vicinity of